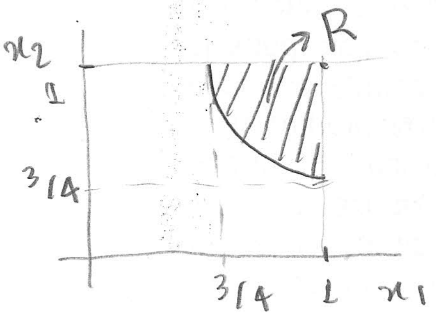
# $2^{a}$ prova

$\boxed{1}$ $R = \{(x_1, x_2) : x_1 x_2 \geq \frac{3}{4}\}$

(a) 


(b) $\beta(\theta) = P_\theta\left(X_1 X_2 \geq \frac{3}{4}\right)$

$$= \int_{3/4}^{1} \int_{\frac{3}{4x_1}}^{1} \theta^2 x_1^{\theta-1} x_2^{\theta-1} \, dx_2 \, dx_1$$

$$= 1 - \left(\frac{3}{4}\right)^\theta + \theta \left(\frac{3}{4}\right)^\theta \log(3/4), \quad \theta > 0.$$

Como $f(x;\theta) = \theta x^\theta / x \cdot I_{(0,1)}(x) = \exp(\theta \log x + \theta - x) I_{(0,1)}(x)$, a dist. é da fam. exp. com $c(\theta) = \theta$, que é monótona crescente. $\Rightarrow$ $\alpha = \beta(1) = 1 - \frac{3}{4} + \frac{3}{4}\log(3/4) \cong 0{,}0342.$

(c) É UMP, pois $\log x_1 + \log x_2 \geq C \Leftrightarrow x_1 x_2 \geq C^*$.

$\boxed{2}$ $Y_i - \beta_0 - \beta_1 X_i \overset{ind}{\sim} N(0, \lambda_i + \beta_1^2 K_i)$, $i = 1, \dots, n$ $\Rightarrow$ $\dfrac{Y_i - \beta_0 - \beta_1 X_i}{\sqrt{\lambda_i + \beta_1^2 K_i}} \overset{iid}{\sim} N(0,1)$, $i = 1, \dots, n$

$$\Rightarrow \sum_{i=1}^{n} \frac{Y_i - \beta_0 - \beta_1 X_i}{\sqrt{\lambda_i + \beta_1^2 K_i}} \sim N(0, n).$$

$\boxed{3}$ $L(\theta) \propto \theta^{2x_1} \{2\theta(1-\theta)\}^{x_2} (1-\theta)^{2x_3}$

$$\propto \theta^{2x_1 + x_2} (1-\theta)^{x_2 + 2x_3},$$

$$l(\theta) = c + (2x_1 + x_2)\log\theta + (x_2 + 2x_3)\log(1-\theta)$$

$\dfrac{\partial l(\theta)}{\partial \theta} = \dfrac{2x_1 + x_2}{\theta} - \dfrac{x_2 + 2x_3}{1-\theta}$; $\dfrac{\partial l(\theta)}{\partial\theta} = 0 \Rightarrow \hat{\theta} = \dfrac{2X_1 + X_2}{2n}$ (EMV de $\theta$).

$$\frac{\partial^2 l(\theta)}{\partial\theta^2} = \frac{-(2x_1 + x_2)}{\theta^2} - \frac{(x_2 + 2x_3)}{(1-\theta)^2} < 0$$

Estatística da RV:

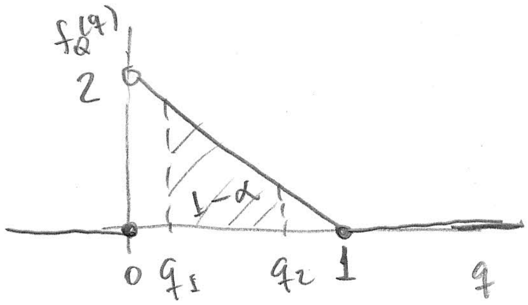
$$\lambda(\underset{\sim}{x}) = \frac{L(\theta_0)}{L(\hat{\theta})} = \frac{\theta_0^{2x_1+x_2} \times (1-\theta_0)^{x_2+2x_3}}{\hat{\theta}^{2x_1+x_2} \times (1-\hat{\theta})^{x_2+2x_3}}$$

$R = \{\underset{\sim}{x} : \lambda(\underset{\sim}{x}) \le c\}$. Teste assintótico:

$\Lambda_n = -2\log(\lambda(\underset{\sim}{x}))$. Rejeitar Ho se, e somente se, $\Lambda_n > \chi^2_{1,crit}$, em que $P(Q > \chi^2_{1,crit}) = \alpha$ e $Q \sim \chi^2_1$.

$\boxed{4.}$ $f(x;\theta) = \frac{2}{\theta}\left(\frac{\theta - x}{\theta}\right) I_{(0,\theta)}(x) = \frac{2}{\theta}\cdot\left(1 - \frac{x}{\theta}\right)\cdot I_{(0,1)}(x/\theta)$

$\Rightarrow Q = \frac{X}{\theta}$ é um pivô com $f_Q(q) = 2(1-q)\, I_{(0,1)}(q)$




Existem $q_1$ e $q_2$ $(q_1 < q_2)$ tais que $P\left(q_1 \le \frac{X}{\theta} \le q_2\right) = 1-\alpha$.

$P\left(\frac{1}{q_2} \le \frac{\theta}{X} \le \frac{1}{q_1}\right) = 1-\alpha \Leftrightarrow P\left(\frac{X}{q_2} \le \theta \le \frac{X}{q_1}\right) = 1-\alpha$

$\Rightarrow \left[\frac{X}{q_2}, \frac{X}{q_1}\right]$ é um IC $100(1-\alpha)\%$ para $\theta$, com amplitude $X\left(\frac{1}{q_1} - \frac{1}{q_2}\right)$. $\frac{1}{q_1} - \frac{1}{q_2}$ deve ser minimizada sujeita a

$2\int_{q_1}^{q_2} (1-q)\,dq = 1-\alpha$, ou seja, $q_2\left(1 - \frac{q_2}{2}\right) - q_1\left(1 - \frac{q_1}{2}\right) = \frac{1-\alpha}{2}$.

$\boxed{5}$ (a) $f(x;\theta_1) = \frac{\theta_2^{\theta_1}}{\Gamma(\theta_1)} x^{\theta_1 - 1} e^{-\theta_2 x} I_{(0,\infty)}(x)$

Fam. exp. com $T(x) = \log x$ e $c(\theta_1) = \theta_1$ (monótona crescente).

teste UMP: $R = \{\underset{\sim}{x} : \sum_{i=1}^{n} \log x_i > c_1\}$

(b) $f(x;\theta_2) = \frac{\theta_2^{\theta_1}}{\Gamma(\theta_1)} x^{\theta_1 - 1} e^{-\theta_2 x} I_{(0,\infty)}(x)$.

Fam. exp. com $T(x) = x$ e $c(\theta_2) = -\theta_2$ (monótona decrescente).

Teste UMP: $R = \{\underset{\sim}{x} : \sum_{i=1}^{n} x_i < c_2\}$