Complex Network Models

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Random Network

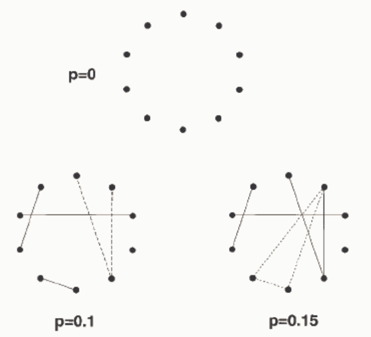


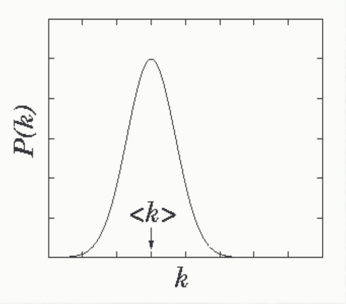
FIG. 5. Illustration of the graph evolution process for the Erdős-Rényi model. We start with $N=10$ isolated nodes (upper panel), then connect every pair of nodes with probability p. The lower panel of the figure shows two different stages in the graph's development, corresponding to $p=0.1$ and $p=0.15$. We can notice the emergence of trees (a tree of order 3, drawn with long-dashed lines) and cycles (a cycle of order 3, drawn with short-dashed lines) in the graph, and a connected cluster that unites half of the nodes at $p=0.15=1.5/N$.

Random Network

$$P(k_i = k) = C_{N-1}^{k} p^k (1-p)^{N-1-k}$$

$$P(k) = e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}$$

Poission distribution



Random Network

$$\lim_{N\to\infty} P_{N,p}(Q) = \begin{cases} 0 & if \quad \dfrac{p(N)}{p_c(N)} \to 0 \\ 1 & if \quad \dfrac{p(N)}{p_c(N)} \to \infty \end{cases}$$

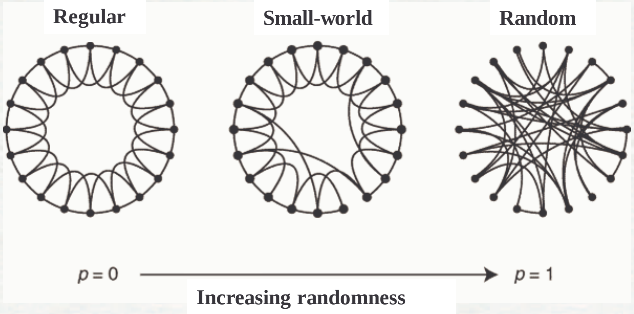
For many such properties, there is a critical property $p_c(N)$. If $p(N)$ grows more slowly than $p_c(N)$, then almost every graph with connection probability $p(N)$ fails to have property Q. If $p(N)$ grows somewhat faster than $p_c(N)$, then almost every graph has Q.

Random Network

A few important special cases are:

1) The critical probability of having a tree of order k is $p_c(N) = cN^{-k/(k-1)}$;
2) The critical probability of having a cycle of order k is $p_c(N) = cN^{-1}$;
3) The critical probability of having a complete subgraph of order k is $p_c(N) = cN^{-2/(k-1)}$;

Small-world network



This process generates $pNK/2$ long-range connections.

D. J. Watts and S. H. Strogatz, *Nature*, **393**, 440–442 (1998).

Small-world network

Average path length

$$\ell = \frac{1}{\frac{1}{2}n(n+1)} \sum_{i \geq j} d_{ij}$$

d_{ij}: shortest path between *i* and *j*.

Clustering Coefficient

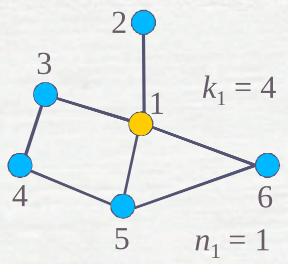
$$C_i(k_i) = \frac{2n_i}{k_i(k_i - 1)}$$

n_i: number of triangles connected to *i*.

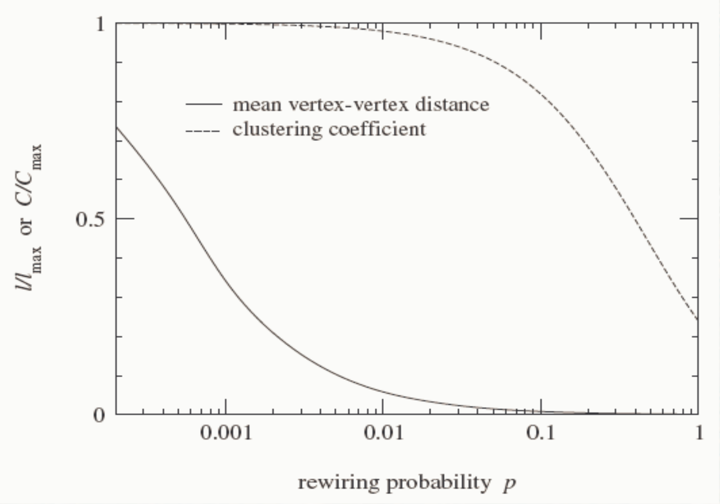
$$C = \frac{1}{n} \sum_i C_i(k_i)$$

Small-world network

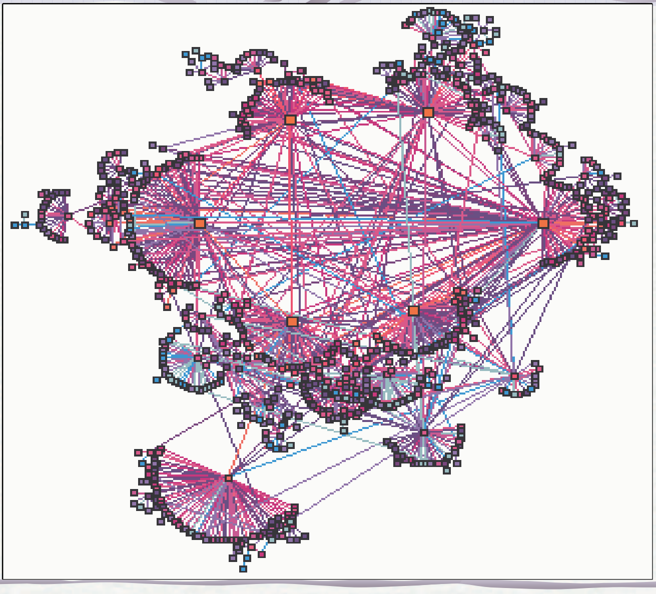
- Example: vertice 1 has 4 neighbors (k_i = 4). Among them, only 1 pair is connected (n_1 = 1). The total number of triangles passing vertice *i* is thus 1. Then, C_1 = 1 / 6.



Small-world network



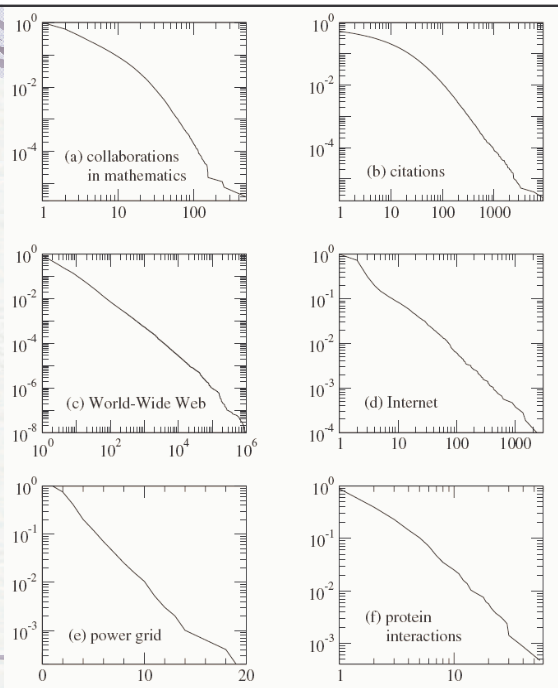
Scale-free Network



Scale-free Network

Degree distribution obeys power law: $P(k) \sim k^{\alpha}$.

A.-L. Barabasi and, R. Albert, *Science* **286**, 509 (1999).



Scale-free Network

(1) Growth: Starting with a small number (m_0) of vertices, at every timestep we add a new vertex with *m* edges (that will be connected to the vertices already present in the system).
(2) Preferential attachment: When choosing the vertices to which the new vertex connects, we assume that the probability that a new vertex will be connected to vertex i depends on the connectivity k_i of that vertex, such that

$$\Pi(k_i) = \frac{k_i}{\sum_j k_j}$$

A.-L. Barabasi and, R. Albert, *Science* **286**, 509 (1999).

Scale-free Network

$$\frac{\partial k_i}{\partial t} = m\Pi(k_i) = m\frac{k_i}{\sum_{j=1}^{m_0+t-1} k_j}$$

Taking into account that $\sum_j k_j = 2mt$

$$\frac{\partial k_i}{\partial t} = \frac{k_i}{2t}$$

Scale-free Network

The solution of this equation, with the initial condition that vertex i was added to the system at time t_i with connectivity $k_i(t_i) = m$, is

$$k_i(t) = m\left(\frac{t}{t_i}\right)^{0.5}$$

Scale-free Network

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$$k_i(t) = m\left(\frac{t}{t_i}\right)^{0.5}$$

$$P(k_i < k) = P\left(t_i > \frac{m^2 t}{k^2}\right)$$

Scale-free Network

Assuming that we add the vertices at equal time intervals to the system, the probability density of t_i is

$$P_i(t_i) = \frac{1}{m_0 + t}$$

$$P(k_i < k) = P\left(t_i > \frac{m^2 t}{k^2}\right) = 1 - P\left(t_i \le \frac{m^2 t}{k^2}\right) = 1 - \frac{m^2 t}{k^2(t + m_0)}$$

Scale-free Network

The probability density for $P(k)$ can be obtained using

$$P(k) = \frac{\partial P(k_i(t) < k)}{\partial k} = \frac{2m^2 t}{m_0 + t} \frac{1}{k^3}$$

Predicting $\gamma = 3$

Random Network

(1) Growth: Starting with a small number of vertices (m_0), at every timestep we add a new vertex with m edges.
(2) Uniform attachment: We assume that the new vertex connects with equal probability to the vertices already present in the system, i.e.

$$\Pi(k_i) = \frac{1}{m_0 + t - 1}$$

Random Network

$$\frac{\partial k_i}{\partial t} = m\Pi(k_i) = \frac{m}{m_0 + t - 1}$$

$$k_i = m(\ln(m_0 + t - 1) - \ln(m_0 + t_\mathrm{i} - 1) + 1)$$

Random Network

The probability that vertex i has connectivity $k_\mathrm{i}(t)$ smaller than k is

$$P(k_i < k) = P\left(t_i > (m_0 + t - 1)\exp\left(1 - \frac{k}{m}\right) - m_0 + 1 \right)$$

$$P\left(t_i > (m_0 + t - 1)\exp\left(1 - \frac{k}{m}\right) - m_0 + 1 \right)$$

$$= 1 - \frac{(m_0 + t - 1)\exp\left(1 - \frac{k}{m}\right) - m_0 + 1}{m_0 + t}$$

Random Network

The probability that vertex i has connectivity $k_i(t)$ smaller than k is

$$P(k) = \frac{e}{m}\exp\left(-\frac{k}{m}\right)$$

Master Equation Approach

We study the probability $p(k, t_i, t)$ *that a vertex i that entered the system at* time t_i *has degree k at time t. During the graph process the degree of a vertex i increases by one with probability* $k/2t$.

A master equation for this probability $p(k, t_i, t)$ *is of the form:*

$$\dot{p}(k, t_i, t) = \sum_{k'} [W_{k' \to k}\, p(k', t_i, t) - W_{k \to k'}\, p(k, t_i, t)]$$

Master Equation Approach

Here $W_{k' \to k}$ denotes the probability of changing from state k' to state k. In our model this probability is obviously

$$W_{k' \to k} = \frac{k'}{2t}\delta_{k',k-1} \qquad , \text{ where } \quad \delta_{i,j} = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases} \tag{13.7}$$

is the Kronecker symbol.

Master Equation Approach

By summing up over all vertices inserted up to time t, we define the probability $P(k,t) := \frac{\sum_{t_i}^{t} p(k,t_i,t)}{t}$ that some arbitrary vertex has degree k.

As we are interested in a stationary distribution, we are looking for the point where the derivative with respect to time is zero.

$$\begin{aligned}
0 = \dot{P}(k,t) &= \frac{t \sum_{t_i} \dot{p}(k,t_i,t) - \sum_{t_i} p(k,t_i,t)}{t^2} \\
&= \left(\frac{1}{t} \sum_{t_i} \dot{p}(k,t_i,t) \right) - \frac{1}{t} P(k,t) \\
&= \left(\sum_{k'} \frac{1}{t} \left[W_{k' \to k}\, p(k',t_i,t) - W_{k \to k'}\, p(k,t_i,t) \right] \right) - \frac{1}{t} P(k,t) \\
&= \left(\sum_{k'} \left[W_{k' \to k}\, P(k',t) - W_{k \to k'}\, P(k,t) \right] \right) - \frac{1}{t} P(k,t) \\
&= \left(\sum_{k'} \left[\frac{k'}{2t} \delta_{k',k-1} P(k',t) - \frac{k}{2t} \delta_{k,k'-1} P(k,t) \right] \right) - \frac{2}{2t} P(k,t) \\
&= \frac{k-1}{2t} P(k-1,t) - \frac{k+2}{2t} P(k,t)
\end{aligned}$$

Master Equation Approach

There is now a t' so that for every time t greater than t' we get the stationary distribution, $\tilde{P}(k)$. This results in the recursive equation $\tilde{P}(k) = \frac{k-1}{k+2}\tilde{P}(k-1)$ for $k \geq m+1$. For the case $k = m$ the probability directly results from the scaling condition of the probability measure: $\tilde{P}(m) = \frac{2}{m+2}$.

This directly yields the power law of the form $\Pr[k] = \frac{2m(m+1)}{k(k+1)(k+2)}$ which converges to the value of the power law found using the continuum theory, $2m^2\gamma^{-3}$.

Rate Equation Approach

We are considering the average number (over all graphs of the state of the process) $N_k(t)$ of vertices that have exactly degree k at time t. Asymptotically we have, by the strong law of large numbers, the following for large t: $N_k(t)/t \sim \Pr[k]$ and $\sum_k kN_k(t)/t \sim 2m$.

If a new vertex enters the network, $N_k(t)$ changes as follows:

$$\Pr[k] = \quad \frac{\partial N_k}{\partial t} = m\frac{(k-1)N_{k-1}(t) - kN_k(t)}{\sum_k kN_k(t)} + \delta_{k,m}. \qquad (13.8)$$

Master Equation Approach

A. Barrat, M. Barthélemy and A. Vespignani, Dynamical process on Complex networks, Cambridge University Press, 2008.

Master Equation Approach

$$\partial_t P(\sigma,t) = \sum_{\sigma'} \left[P(\sigma',t) W(\sigma' \to \sigma) - P(\sigma,t) W(\sigma \to \sigma') \right]$$

σ_i : *state of node i*

$\sigma_i = 1,2,\ldots,\kappa$

A particular configuration of the network at time t is represented by

$$\sigma(t) = (\sigma_1(t), \sigma_2(t), \ldots, \sigma_N(t))$$

$P(\sigma,t)$: Probability of finding the system at time t in a given configuration σ

Master Equation Approach

$$\partial_t P(\sigma,t) = \sum_{\sigma'} \left[P(\sigma',t) W(\sigma' \to \sigma) - P(\sigma,t) W(\sigma \to \sigma') \right]$$

$W(\sigma' \to \sigma)$: Transition rate from one state to another

If the change of state of node *i* is determined only by the local interaction with the nodes directly connected to it and the local dynamics have the same parameters for all nodes, the transition rates can be simplified and read

$$W(\sigma' \to \sigma) = \prod_i w\left(\sigma_i' \to \sigma_i \mid \sigma_j\right)$$

$j \in V(i)$ *j* is a neighbor of *i*

Master Equation Approach

$$\partial_t P(\sigma,t) = \sum_{\sigma'} \left[P(\sigma',t) W(\sigma' \to \sigma) - P(\sigma,t) W(\sigma \to \sigma') \right]$$

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Master Equation Approach

$$\partial_t P(\sigma,t) = \sum_{\sigma'} \left[P(\sigma',t) W(\sigma' \to \sigma) - P(\sigma,t) W(\sigma \to \sigma') \right]$$

Given any function of the state of the system A(σ), it is possible to compute its average value at time t as

$$\langle A(t) \rangle = \sum_{\sigma} A(\sigma) P(\sigma,t)$$

Master Equation Approach - example

Let us consider a very simple system in which each node can be in only two states: σ_i=A and σ_i=B. The dynamics of the system are described by a reaction process of the type A + B → 2B. The transition from A to B is irreversible and occurs with rate β each time a node in thestate A is connected to at least one node in the state B.

$$w(A \to A \mid \sigma_j = A) = w(B \to B \mid \sigma_j = A)$$
$$= w(B \to B \mid \sigma_j = B) = 1$$

$$w(A \to B \mid \sigma_j = B) = \beta$$

Master Equation Approach - example

We use the following quantity:

$$N_A(t) = \sum_{\sigma} \sum_{i} \delta_{\sigma_i, A} P(\sigma, t)$$

$$N_B(t) = \sum_{\sigma} \sum_{i} \delta_{\sigma_i, B} P(\sigma, t)$$

Average number of node in state A or B at time *t*

Master Equation Approach

$$\partial_t N_B(t) = \sum_{\sigma} \sum_{i} \delta_{\sigma_i, B} \partial_t P(\sigma, t)$$

$$= \sum_{i} \sum_{\sigma'} \sum_{\sigma} \left[\delta_{\sigma_i, B} \prod_{k} w(\sigma_k^{'} \to \sigma_k \,|\, \sigma_j^{'}) P(\sigma', t) - \delta_{\sigma_i, B} \prod_{k} w(\sigma_k \to \sigma_k^{'} \,|\, \sigma_j) P(\sigma, t) \right]$$

Master Equation Approach

$$\sum_{\sigma'}\sum_{k} w\left(\sigma_k \to \sigma_k^{'} \mid \sigma_j\right) = 1$$

$$\sum_{\sigma} \delta_{\sigma_i,B} \prod_{k} w\left(\sigma_k^{'} \to \sigma_k \mid \sigma_j^{'}\right) = w\left(\sigma_i^{'} \to \sigma_i = B \mid \sigma_j^{'}\right)$$

We have

$$\partial_t N_B(t) = \sum_{\sigma}\sum_{i} \delta_{\sigma_i,B} \partial_t P(\sigma,t)$$

$$= \sum_{i}\sum_{\sigma'}\left[\prod_{k} w\left(\sigma_i^{'} \to \sigma_i = B \mid \sigma_j^{'}\right) P(\sigma',t)\right] - N_B(t)$$

Master Equation Approach

Supose that the probability for each node to be in the state A or B is $p_A = N_A/N$ and $p_B = N_B/N$.

In addition, neglecting correlations allows us to write

$$P(\sigma',t) = \prod_i p_{\sigma_i^{'}}$$

Then we have

$$w\left(\sigma_i^{'} \to \sigma_i = B \mid \sigma_j^{'}\right) P(\sigma',t)$$

$$= \sum_{\sigma_j^{'}}\left[w\left(\sigma_i^{'} = A \to \sigma_i = B \mid \sigma_j^{'}\right) p_A \prod_{j \in V(i)} p_{\sigma_j^{'}} + w\left(\sigma_i^{'} = B \to \sigma_i = B \mid \sigma_j^{'}\right) p_B \prod_{j \in V(i)} p_{\sigma_j^{'}} \right]$$

Master Equation Approach

Remember that

$$w\left(\sigma_i^{'}=B \rightarrow \sigma_i = B \mid \sigma_j^{'}\right)=1$$
$$w\left(\sigma_i^{'}=A \rightarrow \sigma_i = B \mid \sigma_j^{'}\right)=\beta$$

The latter happens if at least one of the connected nodes j is in the state B. This will happen with probability $1-(1-p_B)^k$, where *k* is the number of neighbors of *i*.

$$\partial_t N_B(t)=\sum_i \left(\beta p_A\left(1-(1-p_B)^k\right)+p_B\right)-N_B(t)$$

Master Equation Approach

By using the expressions of p_A and p_B and summing over all nodes *i* we obtain

$$\partial_t N_B(t)=\beta N_A\left(1-\left(1-{N_B}/{N}\right)^k\right)$$

As a final simplification, we consider $N_B/N \ll 1$ that yeilds the dynamical equation

$$\partial_t N_B(t)=\beta k \frac{N_A N_B}{N}$$