Caminhos mínimos
Parte II

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Baseado nos slides da Profa. Rosane Minghin

Caminho mínimo

- Problema: encontrar o caminho de menor custo (ou o menor caminho) entre dois vértices em um grafo valorado
  – Algoritmo de Djikstra;
    - Uma única origem
  – Algoritmo de Floyd-Warshall.
    - Caminhos mais curtos de todos os pares possíveis

Caminho mínimo

- Grafo dirigido G(V, E) com função peso w: E$\rightarrow\Re$ que mapeia as arestas em pesos.
- Peso (custo) do caminho p = <v0, v1, ..., vk>

$$w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$$

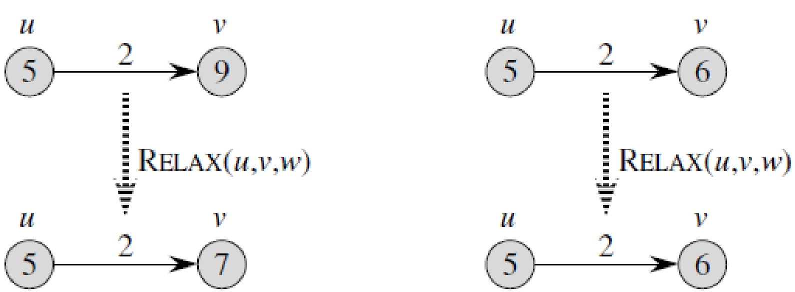
- Custo do caminho de menor peso entre u e v:

$$\delta(u,v) = \begin{cases} \min\{w(p) : u \overset{p}{\Rightarrow} v\} \, se \exists \; rota \; de \; u \;\; p/v \\ \infty \; cc \end{cases}$$

Dijkstra (Cormen, 2001)

```
INITIALIZE-SINGLE-SOURCE(G, s)
for each vertex v ∈ V[G]
    do d[v] ← ∞
       π[v] ← NIL
d[s] ← 0
```

```
RELAX(u, v, w)
if d[v] > d[u] + w(u, v)
   then d[v] ← d[u] + w(u, v)
        π[v] ← u
```



(a) (b)

Algoritmo Dijkstra (Cormen, 2001)

```
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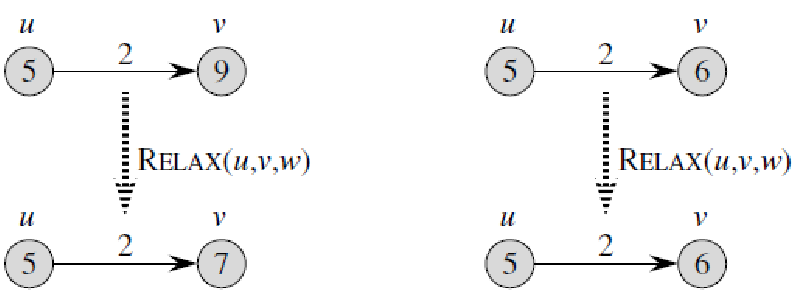
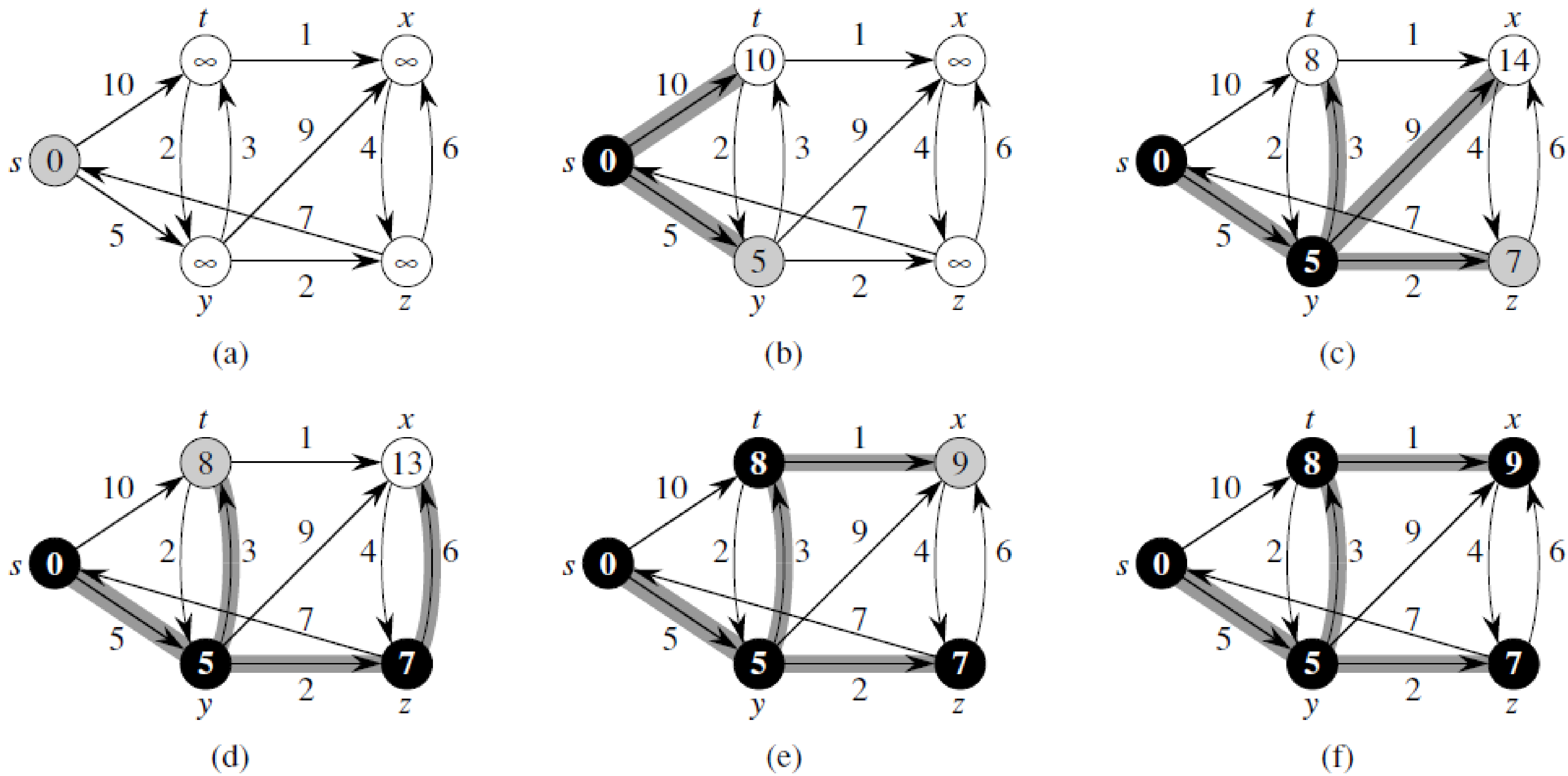


(a) (b)

```
DIJKSTRA(G, w, s)
INITIALIZE-SINGLE-SOURCE(G, s)
S ← Ø
Q ← V[G]
while Q ≠ Ø
      do u ← EXTRACT-MIN(Q)
           S ← S ∪ {u}
           for each vertex v ∈ Adj[u]
                 do RELAX(u, v, w)
```

Exemplo Dijkstra (Cormen, 2001)



Complexidade Dijkstra (Cormen, 2001)

```
DIJKSTRA(G, w, s)
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while Q ≠ Ø
      do u ← EXTRACT-MIN(Q)
         S ← S ∪ {u}
         for each vertex v ∈ Adj[u]
             do RELAX(u, v, w)
```

$O(V)$ (linhas 1–3)

$|V|$ vezes (linhas 4–8)

$grau(u)$ vezes (linhas 7–8)

$O(E)$ DECREASE-KEY's implícitos (RELAX)

Tempo = $V.T_{\text{EXTRACT-MIN}} + E.T_{\text{DECREASE-KEY}}$

Complexidade Dijkstra (Cormen, 2001)

Tempo = $V.T_{\text{EXTRACT-MIN}} + E.T_{\text{DECREASE-KEY}}$

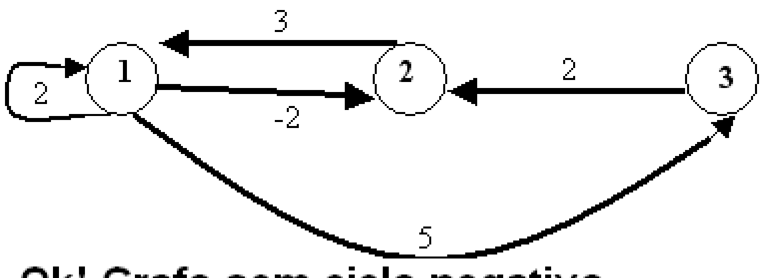
Q	$T_{\text{Extract-Min}}$	$T_{\text{Decrease-Key}}$	Total
array	$O(V)$	$O(1)$	$O(V^2)$
binary heap	$O(\lg V)$	$O(\lg V)$	$O(E \lg V)$

Caminhos entre todos os pares

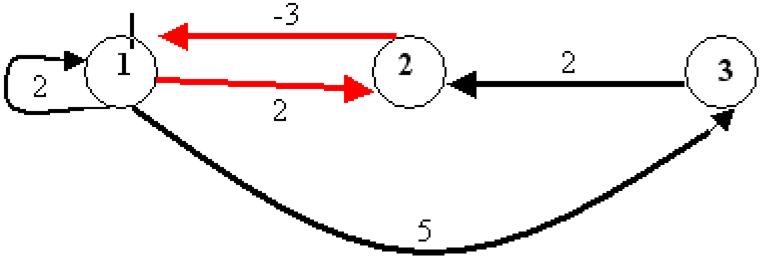
- **Entrada**: Grafo direcionado $G = (V, E)$, com uma função de peso $w : E \rightarrow \mathrm{R}$.
- **Saída**: matriz $n \ x \ n$ com os pesos dos menores caminhos $\delta(i, j)$ entre todos os vértices $i, j \in V$.

Floyd-Warshall

- O algoritmo de Floyd-Warshall determina as distâncias dos menores caminhos entre todos os pares de vértices de um grafo.
- Trabalha com arestas com pesos negativos.
- Mas não funciona quando existem ciclos negativos no grafo.



Ok! Grafo sem ciclo negativo



Nada feito. Grafo com ciclo negativo (arestas vermelhas)

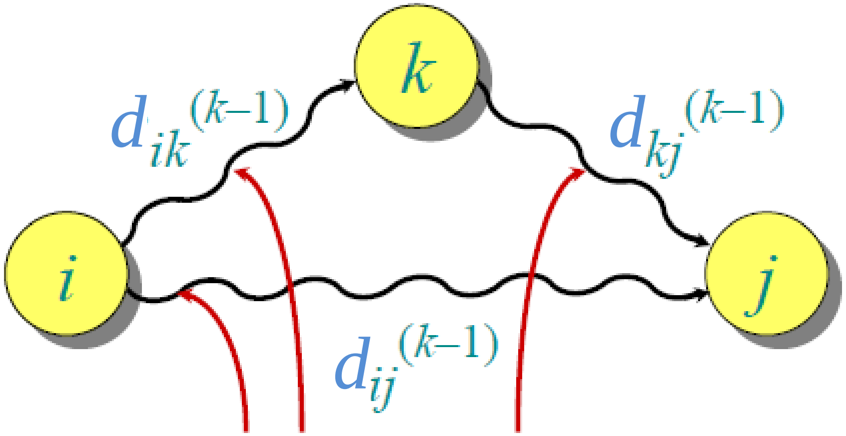
Floyd-Warshall

- Ideia
  - $d_{ij}^{(k)}$: peso do menor caminho do vértice i ao vértice j cujos vértices intermediários pertencem ao conjunto $\{1,2,\ldots,k\}$
  - Começar com $k=0$ e ir atualizando a matriz incrementando o valor de k, ou seja, inserindo mais um vértice no conjunto de vértices intermediários permitidos.

$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0 , \\ \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right) & \text{if } k \geq 1 . \end{cases}$$

Floyd-Warshall

$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0 \, , \\ \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right) & \text{if } k \geq 1 \, . \end{cases}$$



Floyd-Warshall

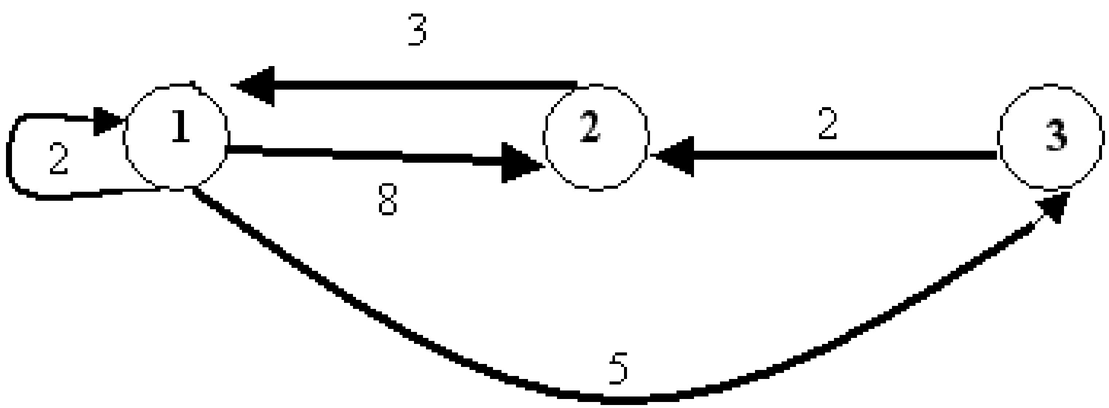
- W: matriz representando os pesos das arestas:

$$w_{ij} = \begin{cases} 0 & \text{if } i = j \,, \\ \text{the weight of directed edge } (i, j) & \text{if } i \neq j \text{ and } (i, j) \in E \,, \\ \infty & \text{if } i \neq j \text{ and } (i, j) \notin E \,. \end{cases}$$

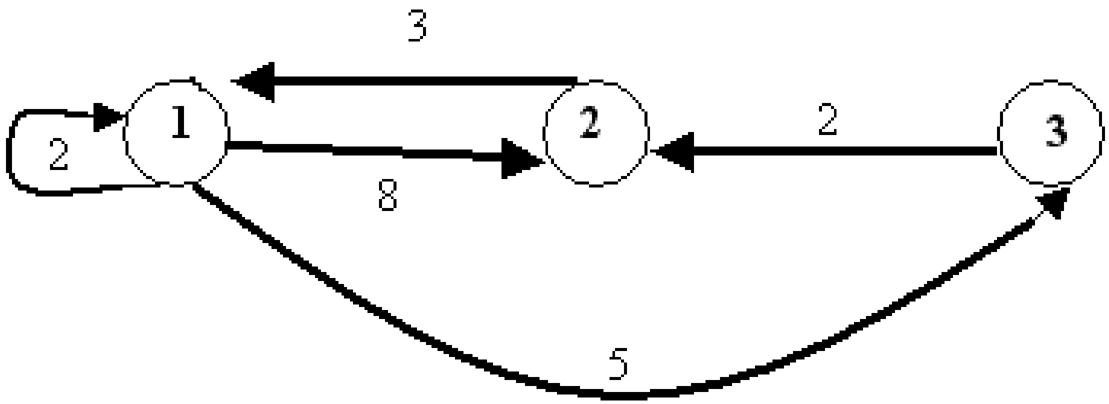
- Algoritmo:

```
FLOYD-WARSHALL(W)
n ← rows[W]
D^(0) ← W
for k ← 1 to n
    do for i ← 1 to n
        do for j ← 1 to n
            do d_ij^(k) ← min(d_ij^(k-1), d_ik^(k-1) + d_kj^(k-1))
return D^(n)
```

Exemplo de Floyd-Warshall



Exemplo de Floyd-Warshall

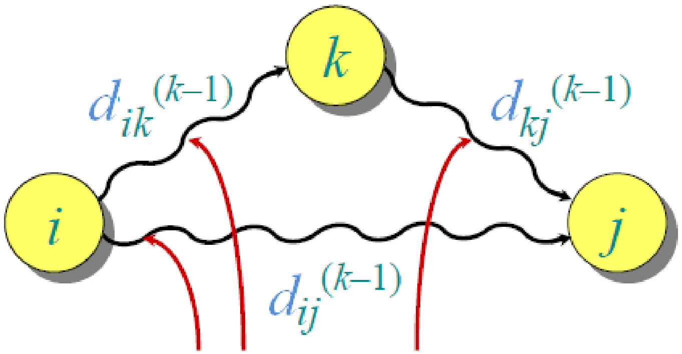


$$D^{(0)} = \begin{bmatrix} 0 & 8 & 5 \\ 3 & 0 & \infty \\ \infty & 2 & 0 \end{bmatrix} \quad D^{(1)} = \begin{bmatrix} 0 & 8 & 5 \\ 3 & 0 & 8 \\ \infty & 2 & 0 \end{bmatrix} \quad D^{(2)} = \begin{bmatrix} 0 & 8 & 5 \\ 3 & 0 & 8 \\ 5 & 2 & 0 \end{bmatrix} \quad D^{(3)} = \begin{bmatrix} 0 & 7 & 5 \\ 3 & 0 & 8 \\ 5 & 2 & 0 \end{bmatrix}$$

Floyd-Warshall

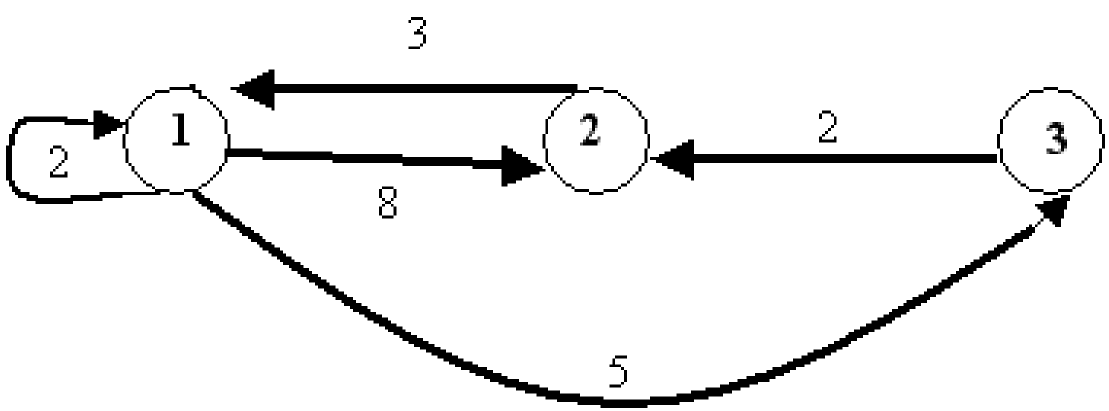
- Caminho?
  - π_{ij}: predecessor do vértice j no menor caminho de i para j com todos os intermediários pertencendo ao conjunto $\{1,2,\ldots,k\}$.

$$\pi_{ij}^{(0)} = \begin{cases} \text{NIL} & \text{if } i = j \text{ or } w_{ij} = \infty , \\ i & \text{if } i \neq j \text{ and } w_{ij} < \infty . \end{cases}$$



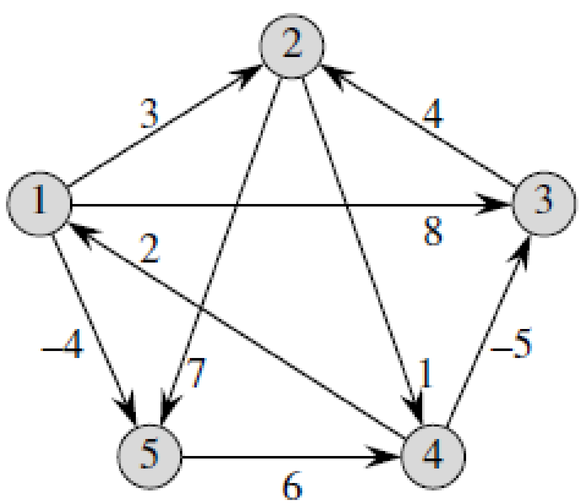
$$\pi_{ij}^{(k)} = \begin{cases} \pi_{ij}^{(k-1)} & \text{if } d_{ij}^{(k-1)} \leq d_{ik}^{(k-1)} + d_{kj}^{(k-1)} , \\ \pi_{kj}^{(k-1)} & \text{if } d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)} . \end{cases}$$

Exemplo de Floyd-Warshall
Armazenando caminho



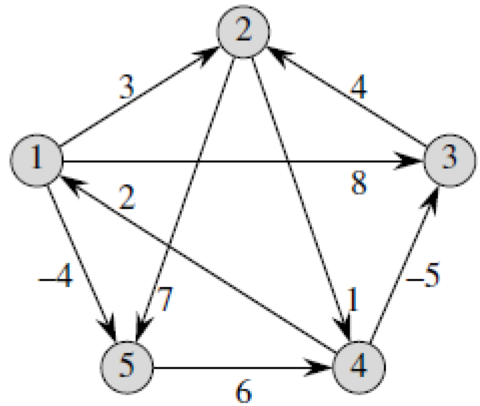
- Quadro...

Exemplo de Floyd-Warshall
Armazenando caminho (Cormen, 2001)



Exemplo de Floyd-Warshall

Armazenando caminho (Cormen, 2001)



$$D^{(0)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \quad \Pi^{(0)} = \begin{pmatrix} \text{NIL} & 1 & 1 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & \text{NIL} & \text{NIL} \\ 4 & \text{NIL} & 4 & \text{NIL} & \text{NIL} \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \quad \Pi^{(1)} = \begin{pmatrix} \text{NIL} & 1 & 1 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & \text{NIL} & \text{NIL} \\ 4 & 1 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \quad \Pi^{(2)} = \begin{pmatrix} \text{NIL} & 1 & 1 & 2 & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & 2 & 2 \\ 4 & 1 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \quad \Pi^{(3)} = \begin{pmatrix} \text{NIL} & 1 & 1 & 2 & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & 2 & 2 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(4)} = \begin{pmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \quad \Pi^{(4)} = \begin{pmatrix} \text{NIL} & 1 & 4 & 2 & 1 \\ 4 & \text{NIL} & 4 & 2 & 1 \\ 4 & 3 & \text{NIL} & 2 & 1 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ 4 & 3 & 4 & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(5)} = \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \quad \Pi^{(5)} = \begin{pmatrix} \text{NIL} & 3 & 4 & 5 & 1 \\ 4 & \text{NIL} & 4 & 2 & 1 \\ 4 & 3 & \text{NIL} & 2 & 1 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ 4 & 3 & 4 & 5 & \text{NIL} \end{pmatrix}$$