Caminhos mínimos
Parte II

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Baseado nos slides da Profa. Rosane Minghin

Complexidade Dijkstra
(Cormen, 2001; MIT OpenCourseWare)

DIJKSTRA(G, w, s)

$O(V)$:
- INITIALIZE-SINGLE-SOURCE(G, s)
- $S \leftarrow \emptyset$
- $Q \leftarrow V[G]$

$|V|$ vezes:
- **while** $Q \neq \emptyset$
- **do** $u \leftarrow$ EXTRACT-MIN(Q)
- $S \leftarrow S \cup \{u\}$

$grau(u)$ vezes:
- **for** each vertex $v \in Adj[u]$
- **do** RELAX(u, v, w)

$O(E)$ DECREASE-KEY's implícitos

Tempo = $V.T_{\text{EXTRACT-MIN}} + E.T_{\text{DECREASE-KEY}}$

Complexidade Dijkstra
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Tempo = $V.T_{\text{EXTRACT-MIN}} + E.T_{\text{DECREASE-KEY}}$

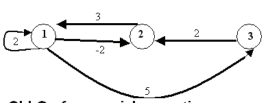
Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
array	$O(V)$	$O(1)$	$O(V^2)$
binary heap	$O(\lg V)$	$O(\lg V)$	$O(E \lg V)$

Caminhos entre todos os pares

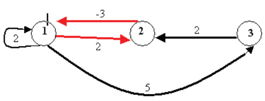
- **Entrada**: Grafo direcionado $G = (V, E)$, com uma função de peso $w : E \rightarrow \mathbb{R}$.
- **Saída**: matriz $n \times n$ com os pesos dos menores caminhos $\delta(i, j)$ entre todos os vértices $i, j \in V$.

Floyd-Warshall

- O algoritmo de Floyd-Warshall determina as distâncias dos menores caminhos entre todos os pares de vértices de um grafo.
- Trabalha com arestas com pesos negativos.
- Mas não funciona quando existem ciclos negativos no grafo.



Ok! Grafo sem ciclo negativo



Nada feito. Grafo com ciclo negativo (arestas vermelhas)

Floyd-Warshall

- Ideia
  - $d_{ij}^{(k)}$: peso do menor caminho do vértice i ao vértice j cujos vértices intermediários pertencem ao conjunto $\{1,2,\ldots,k\}$
  - Começar com k=0 e ir atualizando a matriz incrementando o valor de k, ou seja, inserindo mais um vértice no conjunto de vértices intermediários permitidos.

$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0\,, \\ \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right) & \text{if } k \geq 1\,. \end{cases}$$

Floyd-Warshall
(Cormen, 2001; MIT OpenCourseWare)

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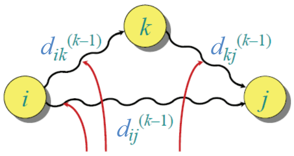


Floyd-Warshall (Cormen, 2001)

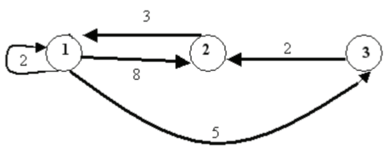
- W: matriz representando os pesos das arestas:

$$w_{ij} = \begin{cases} 0 & \text{if } i = j , \\ \text{the weight of directed edge } (i, j) & \text{if } i \neq j \text{ and } (i, j) \in E , \\ \infty & \text{if } i \neq j \text{ and } (i, j) \notin E . \end{cases}$$

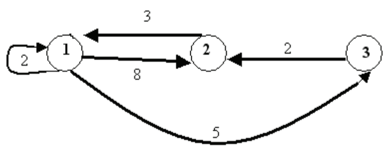
- Algoritmo:

```
FLOYD-WARSHALL(W)
  n ← rows[W]
  D^(0) ← W
  for k ← 1 to n
      do for i ← 1 to n
             do for j ← 1 to n
                    do d_ij^(k) ← min(d_ij^(k-1), d_ik^(k-1) + d_kj^(k-1))
  return D^(n)
```


Exemplo de Floyd-Warshall



Exemplo de Floyd-Warshall

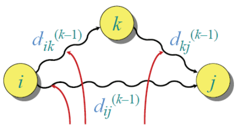


$$D^{(0)} = \begin{bmatrix} 0 & 8 & 5 \\ 3 & 0 & \infty \\ \infty & 2 & 0 \end{bmatrix} \quad D^{(1)} = \begin{bmatrix} 0 & 8 & 5 \\ 3 & 0 & 8 \\ \infty & 2 & 0 \end{bmatrix} \quad D^{(2)} = \begin{bmatrix} 0 & 8 & 5 \\ 3 & 0 & 8 \\ 5 & 2 & 0 \end{bmatrix} \quad D^{(3)} = \begin{bmatrix} 0 & 7 & 5 \\ 3 & 0 & 8 \\ 5 & 2 & 0 \end{bmatrix}$$

Floyd-Warshall
(Cormen, 2001; MIT OpenCourseWare)

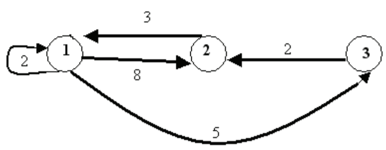
- Caminho?
  - π_{ij}: predecessor do vértice *j* no menor caminho de *i* para *j* com todos os intermediários pertencendo ao conjunto $\{1,2,\ldots,k\}$.

$$\pi_{ij}^{(0)} = \begin{cases} \text{NIL} & \text{if } i = j \text{ or } w_{ij} = \infty , \\ i & \text{if } i \neq j \text{ and } w_{ij} < \infty . \end{cases}$$



$$\pi_{ij}^{(k)} = \begin{cases} \pi_{ij}^{(k-1)} & \text{if } d_{ij}^{(k-1)} \leq d_{ik}^{(k-1)} + d_{kj}^{(k-1)} , \\ \pi_{kj}^{(k-1)} & \text{if } d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)} . \end{cases}$$

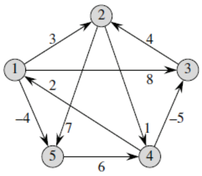
Exemplo de Floyd-Warshall
Armazenando caminho



- Quadro...

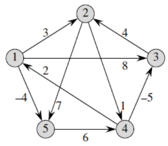
Exemplo de Floyd-Warshall

Armazenando caminho (Cormen, 2001)



Exemplo de Floyd-Warshall

Armazenando caminho (Cormen, 2001)



$$D^{(0)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \quad \Pi^{(0)} = \begin{pmatrix} \text{NIL} & 1 & 1 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & \text{NIL} & \text{NIL} \\ 4 & \text{NIL} & 4 & \text{NIL} & \text{NIL} \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \quad \Pi^{(1)} = \begin{pmatrix} \text{NIL} & 1 & 1 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & \text{NIL} & \text{NIL} \\ 4 & 1 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \quad \Pi^{(2)} = \begin{pmatrix} \text{NIL} & 1 & 1 & 2 & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & 2 & 2 \\ 4 & 1 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \quad \Pi^{(3)} = \begin{pmatrix} \text{NIL} & 1 & 1 & 2 & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & 2 & 2 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(4)} = \begin{pmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \quad \Pi^{(4)} = \begin{pmatrix} \text{NIL} & 1 & 4 & 2 & 1 \\ 4 & \text{NIL} & 4 & 2 & 1 \\ 4 & 3 & \text{NIL} & 2 & 1 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ 4 & 3 & 4 & 5 & \text{NIL} \end{pmatrix}$$

$$D^{(5)} = \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \quad \Pi^{(5)} = \begin{pmatrix} \text{NIL} & 3 & 4 & 5 & 1 \\ 4 & \text{NIL} & 4 & 2 & 1 \\ 4 & 3 & \text{NIL} & 2 & 1 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ 4 & 3 & 4 & 5 & \text{NIL} \end{pmatrix}$$