$\boxed{4}$ (a) $f(\underset{\sim}{x};\underset{\sim}{\theta}) = \alpha^n \beta^{n\alpha} \cdot \left(\prod_{i=1}^{n} x_i\right)^{\alpha} \left(\prod_{i=1}^{n} x_i\right)^{-1} \prod_{i=1}^{n} I_{(0,\frac{1}{\beta}]}(x_i)$

$$= \underbrace{\alpha^n \beta^{n\alpha} \cdot \left(\prod_{i=1}^{n} x_i\right)^{\alpha} \cdot I_{(0,\frac{1}{\beta}]}(x_{(n)})}_{g(t_1,t_2,\underset{\sim}{\theta})} \cdot \underbrace{\left(\prod_{i=1}^{n} x_i\right)^{-1}}_{h(\underset{\sim}{x})}$$

$\left(\prod_{i=1}^{n} X_i, X_{(n)}\right)^T$ é suficiente.

(b) $L(\underset{\sim}{\theta}) \propto \alpha^n \beta^{n\alpha} \left(\prod_{i=1}^{n} x_i\right)^{\alpha} \cdot I_{(0,\frac{1}{x_{(n)}}]}(\beta)$

Para todo $\alpha > 0$, $\alpha^n \beta^{n\alpha} \left(\prod_{i=1}^{n} x_i\right)^{\alpha}$ é crescente em β.

Logo, $\hat{\beta} = 1/X_{(n)}$.

Substituindo $\hat{\beta}$,

$$\ell(\alpha) = c_0 + n \cdot \log(\alpha) + n\alpha \log(\hat{\beta}) + \alpha \sum_{i=1}^{n} \log(x_i)$$

$$\frac{\partial \ell(\alpha)}{\partial \alpha} = \frac{n}{\alpha} + n \log(\hat{\beta}) + \sum_{i=1}^{n} \log(x_i)$$

$$\frac{\partial \ell(\alpha)}{\partial \alpha} = 0 \Rightarrow \alpha = -\frac{1}{\log(\hat{\beta}) + \overline{\log(x)}} = \frac{1}{\log(x_{(n)}) - \overline{\log(x)}}$$

$$\frac{\partial^2 \ell(\alpha)}{\partial \alpha^2} = -\frac{n}{\alpha^2} < 0 \Rightarrow \hat{\alpha} = \frac{1}{\log(X_{(n)}) - \overline{\log(X)}}.$$