Neurodynamics

Dynamical System:

A dynamical system may be defined as a deterministic mathematical prescription for evolving the state of a system forward in time

Continuous System: Time is a continuous variable

$$\frac{dx_1}{dt} = F_1(x_1, x_2, ..., x_N)$$

$$\frac{dx_2}{dt} = F_2(x_1, x_2, ..., x_N) \quad \text{or} \quad \dot{\mathbf{x}} = F(\mathbf{x}, \boldsymbol{\mu}) \quad \boldsymbol{x} \in \Re^N$$

......

$$\frac{dx_N}{dt} = F_N(x_1, x_2, ..., x_N)$$

With $\mathbf{x} \in U \subset \Re^N$, $\boldsymbol{\mu} \in V \subset \Re^P$
where U and V are open sets in $\Re^N$ and $\Re^P$

Neurodynamics

Discrete System: Time is a discrete integer-valued variable

$$\mathbf{x}_{N+1} = F(\mathbf{x}_N)$$

where $\mathbf{x}$ is a N dimensional vector. Given an initial state $\mathbf{x}_0$, we can generate an orbit of the discrete time system $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ...$ by iterating the map.

For example: $x_{N+1} = -0.5x_N$

Continuous Dynamical Systems

For the dynamical system:

$$\dot{\mathbf{x}} = F(\mathbf{x}, \mu) \quad x \in \Re^N \tag{1}$$

Let $$\mathbf{y} = \mathbf{x} - \mathbf{x}_f(t) \tag{2}$$

Where $x_f(t)$ is a fixed point.

Then, $$\dot{\mathbf{x}} = \dot{\mathbf{x}}_f(t) + \dot{\mathbf{y}}$$

Using the Taylor expanding about $\boldsymbol{x}_f(t)$ gives

$$\dot{\mathbf{x}} = F(\mathbf{x}_f(t)) + DF(\mathbf{x}_f(t))\mathbf{y} + O(|\mathbf{y}|^2) \tag{3}$$

Using the factor that $\dot{\mathbf{x}}_f(t) = F(\mathbf{x}_f(t))$, we get

$$\dot{\mathbf{y}} = DF(\mathbf{x}_f(t))\mathbf{y} + O(|\mathbf{y}|^2) \tag{4}$$

Continuous Dynamical Systems

Since **y** is small, so it is reasonable that the stability of the original system could be answered by studying the associated linear system:

$$\dot{\mathbf{y}} = DF(\mathbf{x}_f(t))\mathbf{y} \tag{5}$$

Then, $$\mathbf{y}(t) = e^{DF(\mathbf{x}_f)t}\mathbf{y}_0 \tag{6}$$

Since $DF(\mathbf{x}_f(t)) = DF(\mathbf{x}_f)$, then

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} \tag{7}$$

Continuous Dynamical Systems

A is called a Jacobian matrix (partial derivatives):

$$\mathbf{A} = DF(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \dots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}) \\ \dots & \dots & \dots \\ \frac{\partial f_n}{\partial x_1}(\mathbf{x}) & \dots & \frac{\partial f_n}{\partial x_n}(\mathbf{x}) \end{bmatrix}$$

Continuous Dynamical Systems

We have an eigenvalue equation:

$$\mathbf{Ae} = \lambda \mathbf{e} \qquad (8)$$

$$\mathbf{y}(t) = \sum_{k=1}^{N} A_k e_k \exp(\lambda_k t) \qquad (9)$$

where $\mathbf{A}_k$ are determined from the initial condition: $y_0 = \sum_{k=1}^{N} A_k e_k$

λ_k (k = 1, 2, ..., N) are eigenvalues of **A**

Continuous Dynamical Systems

Definition 1: (*Lyapunov stability*) $\mathbf{x}_f(t)$ is said to be stable or Lyapunov stable, if given $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that, for any other solution, $\mathbf{y}(t)$, of the system (2-2), satisfying $|\mathbf{x}_f(t_0) - \mathbf{y}(t_0)| < \delta$, then $|\mathbf{x}_f(t) - \mathbf{y}(t)| < \varepsilon$ for $t > t_0$, $t_0 \in \Re$.

Definition 2: (*asymptotic stability*) $\mathbf{x}_f(t)$ is said to be asymptotic stable if it is lyapunov stable and if there exists a constant $b > 0$ such that if $|\mathbf{x}_f(t_0) - \mathbf{y}(t_0)| < b$, then $|\mathbf{x}_f(t) - \mathbf{y}(t)| = 0$ when $t \to \infty$.

Definition 3: Let $\mathbf{x} = \mathbf{x}_f$ be a fixed point of = $F(\mathbf{x})$, then, $\mathbf{x}_f$ is called a *hyperbolic* fixed point if none of the eigenvalues of $DF(\mathbf{x}_f)$ have zero real part. (a hyperbolic fixed point of a N-dimensional map is defined as: none of its eigenvalues have absolute one)

Stability of Equilibrium State

Theorem 1: Suppose all of the eigenvalues of **A** in Eqn. (7)
have negative real parts. Then the equilibrium solution $\mathbf{x}_f$ of the nonlinear flow defined by Eqn.(1) is asymptotically stable.

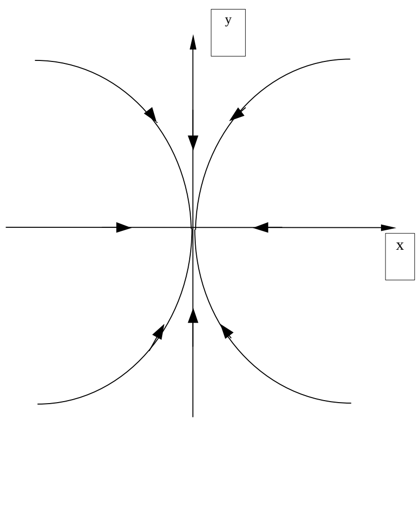
Theorem 2: if one of the eigenvalues of **A** has a positive real part, the fixed point is unstable

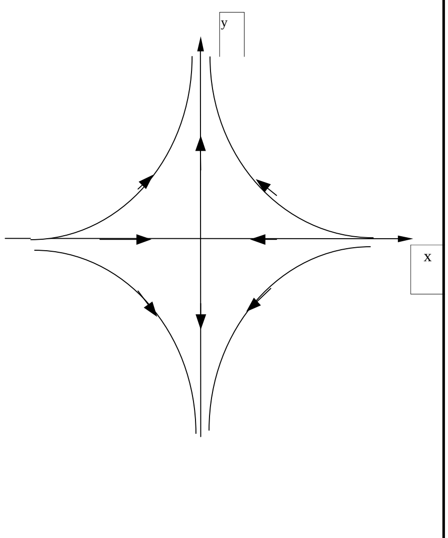
Stability of Equilibrium State

For a two-dimensional continuous system, the stability of a fixed point can be classified by its eigenvalues of the Jacobian matrix evaluated at a fixed point in three types:

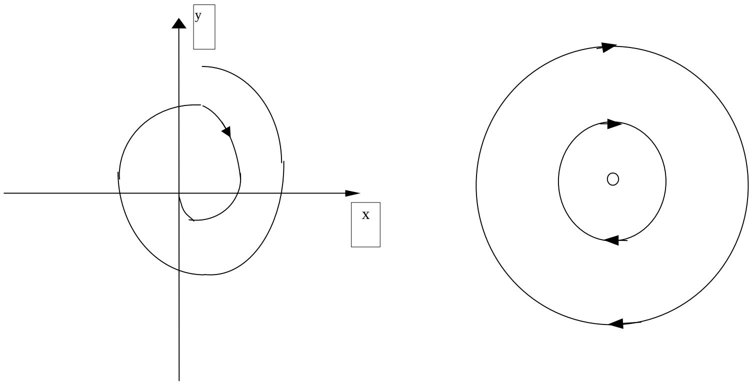
• The eigenvalues are real and have the same sign. If the eigenvalues are negative, this is a *stable point*; if the eigenvalues are positive, this is a *unstable point*.
• The eigenvalues are real and have opposite signs. In this case, there is a one-dimensional stable manifold and a one-dimensional unstable manifold. This fixed point is called a unstable *saddle point*.
• The eigenvalues are complex conjugates. If the real part of the eigenvalues is negative, this is a *stable spiral* or a *spiral sink*. If the real part is positive, this is an *unstable spiral* or a *spiral source*. If the real parts are zero, then this is a *center*.

Stability of Equilibrium State





Stability of Equilibrium State



Stability of Equilibrium State

$$dx/dt = x - x^3/3 - y + I(t)$$
$$dy/dt = c(x + a - by)$$

$(a = 0.7,\ b = 0.8,\ c = 0.1)$

$$J = \begin{bmatrix} 1 - x^2 & -1 \\ c & -bc \end{bmatrix}$$

$$\lambda_{1,\,2} = -[(bc - 1 + x^2) \pm ((x^2 - 1 + bc)^2 - 4c)^{1/2}]\ /2$$

Stability of Equilibrium State

•For $0 < A_0 < 0.341$, the eigenvalues are complex and the real part is negative and hence the fixed point (x_p, y_p) is a stable focus.
•At $A_0 = 0.341$, the real part of eigenvalues vanishes and the system undergoes a Hopf bifurcation.
•For $0.341 < A_0 < 1.397$, the real part is positive and the fixed point is unstable, one can expect a stable limitcycle in this range.
•At $A_0 = 1.397$ the real part of the eigenvalues vanishes and the system undergoes Hopf bifurcation.
•For $A_0 > 1.397$, the real part is negative, the system's solution is again a stable fixed point.
•For sufficiently large A_0, the system points diverge to infinity.

Stability of Equilibrium State

Lyapunov Function

According to the definition of stability it would be sufficient to find a neighborhood U of $\mathbf{x}_f$ for which orbits starting in U remain in U for positive times. This condition would be satisfied if we could show that the vector field is either tangent to the boundary of U or pointing inward toward $\mathbf{x}_f$.

Stability of Equilibrium State

Lyapunov Function

Theorem: Consider the following continuous dynamical system

$$\dot{\mathbf{x}} = F(\mathbf{x}) \qquad \mathbf{x} \in \Re^N$$

Let $\mathbf{x}_f$ be a fixed point and let V: $U \rightarrow \Re$ be a continuous differentiable function defined on some neighborhood U of $\mathbf{x}_f$ such that

i) $V(\mathbf{x}_f) = 0$ and $V(\mathbf{x}) > 0$ if $\mathbf{x} \neq \mathbf{x}_f$.

ii) $\dot{V}(\mathbf{x}) \leq 0$ in $U - \{\mathbf{x}_f\}$.

Then $\mathbf{x}_f$ is stable. Moreover, if

iii) $\dot{V}(\mathbf{x}) < 0$ in $U - \{\mathbf{x}_f\}$.

Then $\mathbf{x}_f$ is asymptotically stable.

Stability of Equilibrium State

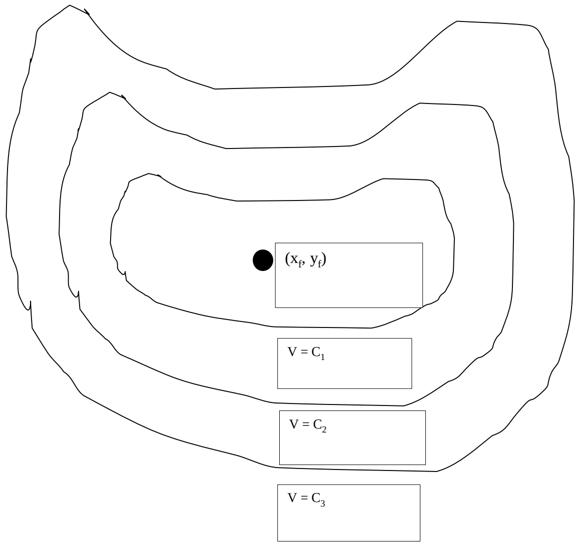
Lyapunov Function

$$\dot{x} = f(x, y) \qquad \dot{y} = g(x, y) \qquad (x, y) \in \Re^2$$

Let (x_f, y_f) be a fixed point. Let $V(x, y)$ be the scalar-valued function, V: $\Re^2 \rightarrow \Re^1$, with $V(x_f, y_f) = 0$, and the locus of points satisfying $V(x, y) = C =$ *constant* form closed curves for different values of C encircling (x_f, y_f) with $V(x, y) > 0$ in a neighborhood of (x_f, y_f).

Stability of Equilibrium State

Lyapunov Function



Stability of Equilibrium State

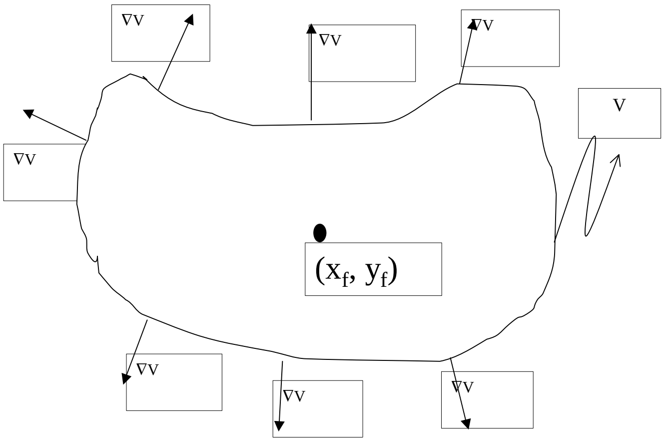
Lyapunov Function

The gradient of *V*, ∇*V*, is a vector perpendicular to the tangent vector along each curve of *V* which points in the direction of increasing *V*. So if the vector field were always to be either tangent or pointing inward for each of these curves surrounding (x_f, y_f), we would have

$$\nabla V(x,\ y) \cdot \left(\dot{x}, \dot{y} \right) \leq 0$$

Stability of Equilibrium State

Lyapunov Function



Discrete Dynamical Systems

Linearized Stability of Equilibrium Solutions

fixed point (*stationary point* or *equilibrium point*):
is an equilibrium solution of a point $\mathbf{x}_f \in \Re^N$ such that
$\mathbf{x}_f = F(\mathbf{x}_f(t))$.

Example 1 (logistic map) : $x_{n+1} = f(x_n) = x_n(1-x_n)$

There is only one fixed point: $x = 0$

Example 2 (logistic map): $x_{n+1} = f(x_n) = 2x_n(1-x_n)$

There are two fixed points: $x = 0$ and $x = 0.5$

Discrete Dynamical Systems

Linearized Stability of Equilibrium Solutions

Example 3 (Henon map): $$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a - x_n^2 + by_n \\ x_n \end{pmatrix}$$

a = 1, b = 1

There are two fixed points: (x = 1, y = 1) and (x = -1, y = -1)

Stability of Equilibrium State

$$\mathbf{x}_N = \mathbf{x}_f + \mathbf{y}_N \quad (10)$$

$$\mathbf{y}_{N+1} = DF(\mathbf{x}_f)\mathbf{y}_N + O(\mathbf{y}_N^2) \quad (11)$$

$$\mathbf{y}_{N+1} = \mathbf{A}\mathbf{y}_N \quad (12)$$

$$\mathbf{A}\mathbf{e} = \lambda\mathbf{e} \quad (13)$$

$$y_{n+1} = \sum_{k=1}^{N} A_k e_k \lambda_k^n \quad (14)$$

Conclusion:
directions corresponding to $|\lambda_k| > 1$ are unstable;
directions corresponding to $|\lambda_k| < 1$ is stable.

Discrete Dynamical Systems

Linearized Stability of Equilibrium Solutions

Example 1 (logistic map) : $x_{n+1} = f(x_n) = x_n(1-x_n)$

There is only one fixed point: x = 0

Since $f'(x)|_{x=0} = 1 - 2x|_{x=0} = 1$, then x = 0 is a center

Example 2 (logistic map): $x_{n+1} = f(x_n) = 2x_n(1-x_n)$

There are two fixed points: x = 0 and x = 0.5

Since $f'(x)|_{x=0} = 2 - 4x|_{x=0} = 2$, then x = 0 is unstable

Since $f'(x)|_{x=0.5} = 2 - 4x|_{x=0.5} = 0$, then x = 0 stable (super-stable)

Discrete Dynamical Systems

Linearized Stability of Equilibrium Solutions

Example 3 (Henon map): $$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a - x_n^2 + by_n \\ x_n \end{pmatrix}$$

a = 1, b = 1

There are two fixed points: (x = 1, y = 1) and (x = -1, y = -1)

$$DF(\mathbf{x}) = \begin{bmatrix} -2x & 1 \\ 1 & 0 \end{bmatrix}$$

For (x=1 , y = 1) $\lambda_1 = -1 + \sqrt{2}$, $\lambda_2 = -1 - \sqrt{2}$, unstable

For (x=-1 , y = -1) $\lambda_1 = 1 + \sqrt{2}$, $\lambda_2 = 1 - \sqrt{2}$, unstable

Stability of Equilibrium State

Exercises:

1) Let l(x) = ax+b, where a and b are constants. Foe which values of a and b does l have na attracting fixed point? A repelling fixed point?

2) Let f(x) = x- x^2. Show that x = 0 is a fixed point of f, and described the dynamical behavior of points near 0.

3) For each of the following linear maps, decide whether the origin is a sink, source, or saddle.

$$\begin{pmatrix} 4 & 30 \\ 1 & 3 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1/2 \\ 1/4 & 3/4 \end{pmatrix} \qquad \begin{pmatrix} -0.4 & 2.4 \\ -0.4 & 1.6 \end{pmatrix}$$

Stability of Equilibrium State

Exercises:

4) Let g(x, y) = (x^2-5x+y, x^2). Find and classify the fixed points of g as sink, source, or saddle.

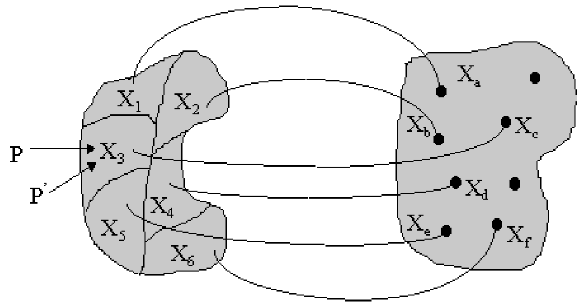
5) Let f(x, y) = (sin(π/3)x, y/2). Find all fixed points and their stability. Where does the orbit of each initial value go?

6) Find

$$\lim_{n\to\infty}\begin{pmatrix} 4.5 & 8 \\ -2 & -3.5 \end{pmatrix}^n \begin{pmatrix} 6 \\ 9 \end{pmatrix}$$

Hopfield Models

General Idea: Artificial Neural Networks ↔ Dynamical Systems



Continuous Hopfield Model

$$C_i \frac{dx_i(t)}{dt} = -\frac{x_i(t)}{R_i} + \sum_{j=1}^{N} w_{ij}\varphi_j(x_j(t)) + I_i$$

a) the synaptic weight matrix is symmetric, $w_{ij} = w_{ji}$, for all i and j.

b) Each neuron has a nonlinear activation of its own, i.e. $y_i = \varphi_i(x_i)$. Here, $\varphi_i(\bullet)$ is chosen as a sigmoid function;

c) The inverse of the nonlinear activation function exists, so we may write $x = \varphi_i^{-1}(y)$.

Continuous Hopfield Model

Lyapunov Function:

$$E = -\frac{1}{2}\sum_{j=1}^{N}\sum_{i=1}^{N} w_{ij} y_i y_j + \sum_{i=1}^{N}\frac{1}{R_i}\int_0^{x_i}\varphi_i^{-1}(y_i)dx - \sum_{i=1}^{N} I_i y_i$$

$$\frac{dE}{dt} = -\sum_{i=1}^{N}\left(\sum_{j=1}^{N} w_{ij} y_j - \frac{x_i}{R_i} + I_i\right)\frac{dy_i}{dt}$$

$$= -\sum_{i=1}^{N} C_i \left[\frac{d\varphi_i^{-1}(y_i)}{dt}\right]\frac{dy_i}{dt}$$

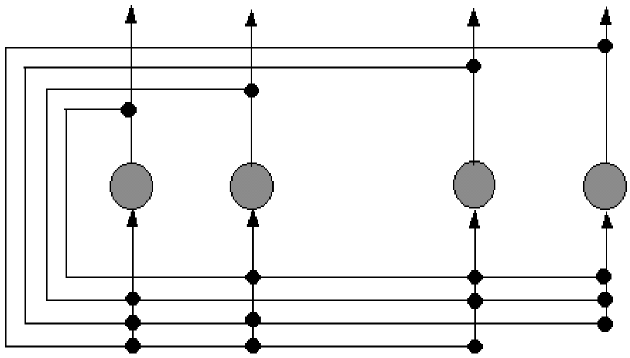
$$= -\sum_{i=1}^{N} C_i \left(\frac{dy_i}{dt}\right)^2 \left[\frac{d\varphi_i^{-1}(y_i)}{dt}\right]$$

$$\leq 0$$

Discrete Hopfield Model

- Recurrent network
- Fully connected
- Symmetrically connected ($w_{ij} = w_{ji}$, or $W = W^T$)
- Zero self-feedback ($w_{ii} = 0$)
- One layer
- Binary States:
  - $x_i = 1$ firing at maximum value
  - $x_i = 0$ not firing
- or Bipolar
  - $x_i = 1$ firing at maximum value
  - $x_i = -1$ not firing

Discrete Hopfield Model



Discrete Hopfield Model (Bipole)

Transfer Function for Neuron *i*:

$$x_i = \begin{cases} 1 & \sum_{j \neq i} w_{ij} x_j - \theta_i > 0 \\ -1 & \sum_{j \neq i} w_{ij} x_j - \theta_i < 0 \\ x_i & \sum_{j \neq i} w_{ij} x_j - \theta_i = 0 \end{cases}$$

x *= (x_1, x_2 ... x_N): bipole vector, network state.*

θ_i : threshold value of x_i.

$$x_i = \text{sgn}\left(\sum_{j \neq i} w_{ij} x_j - \theta_i \right) \qquad \mathbf{x} = \text{sgn}\left(\mathbf{W}\mathbf{x} - \Theta\right)$$

Discrete Hopfield Model

Energy Function:

$$E = -\frac{1}{2}\sum_i \sum_{j \neq i} w_{ij} x_i x_j + \sum_i \theta_i x_i$$

For simplicity, we consider all threshold $\theta_i = 0$:

$$E = -\frac{1}{2}\sum_i \sum_{j \neq i} w_{ij} x_i x_j$$

Discrete Hopfield Model

Learning Prescription (Hebbian Learning):

$$w_{ij} = \frac{1}{N}\sum_{\mu=1}^{M} \xi_{\mu,i}\xi_{\mu,j}$$

$\{\xi_\mu \mid \mu = 1, 2, ..., M\}$: M memory patterns

Pattern $\xi^s = (\xi^s_1, \xi^s_2, ..., \xi^s_n)$, where ξ^s_i take value 1 or -1

In the matrix form:

$$\mathbf{W} = \frac{1}{N}\sum_{\mu=1}^{M} \boldsymbol{\xi}_\mu \boldsymbol{\xi}_\mu^T - M\mathbf{I}$$

Discrete Hopfield Model

Energy function is lowered by this learning rule:

$$E = -\frac{1}{2}\sum_i \sum_{j \neq i} w_{ij} x_i x_j = -\frac{1}{2}\sum_i \sum_{j \neq i} w_{ij} \xi_{\mu,i} \xi_{\mu,j}$$

$$\Leftrightarrow -\frac{1}{2}\sum_i \sum_{j \neq i} \xi_{\mu,i}^2 \xi_{\mu,j}^2$$

Discrete Hopfield Model

Pattern Association (asynchronous update):

$$\Delta_k E = E(k+1) - E(k)$$

$$= -\frac{1}{2}\sum_i \sum_{j \neq i} w_{ij} x_i (k+1) x_j + \frac{1}{2}\sum_i \sum_{j \neq i} w_{ij} x_i (k) x_j$$

$$\Leftrightarrow -\Delta x_i (k) \sum_{j \neq i} w_{ij} x_j$$

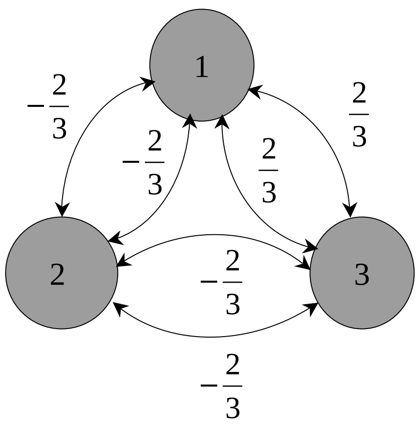
$$\mathbf{\Delta E_k \leq 0}$$

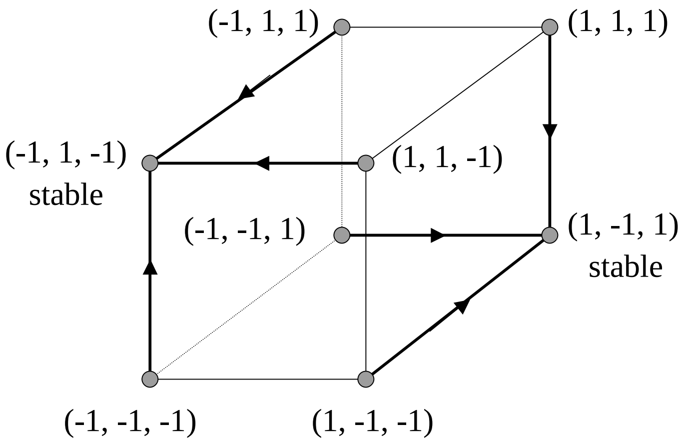
Discrete Hopfield Model

Example:

Consider a network with three neurons, the weight matrix is:

$$\mathbf{W}=\frac{1}{3}\begin{bmatrix} 0 & -2 & 2 \\ -2 & 0 & -2 \\ 2 & -2 & 0 \end{bmatrix}$$





The model with three neurons has two fundamental memories $(-1, 1, -1)^T$ and $(1, -1, 1)^T$

State $(1, -1, 1)^T$:

$$\mathbf{Wx} = \frac{1}{3}\begin{bmatrix} 0 & -2 & 2 \\ -2 & 0 & -2 \\ 2 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 4 \\ -4 \\ 4 \end{bmatrix}$$

$$\operatorname{sgn}[\mathbf{Wx}] = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \mathbf{x}$$

A stable state

State (-1, 1, -1)^{T}:

$$\mathbf{Wx}=\frac{1}{3}\begin{bmatrix} 0 & -2 & 2 \\ -2 & 0 & -2 \\ 2 & -2 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}=\frac{1}{3}\begin{bmatrix} -4 \\ 4 \\ -4 \end{bmatrix}$$

$$\mathrm{sgn}[\mathbf{Wx}]=\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}=\mathbf{x}$$

A stable state

State (1, 1, 1)^{T}:

$$\mathbf{Wx}=\frac{1}{3}\begin{bmatrix} 0 & -2 & 2 \\ -2 & 0 & -2 \\ 2 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}=\frac{1}{3}\begin{bmatrix} 0 \\ -4 \\ 0 \end{bmatrix}$$

$$\mathrm{sgn}[\mathbf{Wx}]=\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\neq\mathbf{x}$$

An unstable state. However, it converges to its nearest stable state (1, -1, 1)^{T}

State (-1, 1, 1)^T:

$$\mathbf{Wx} = \frac{1}{3}\begin{bmatrix} 0 & -2 & 2 \\ -2 & 0 & -2 \\ 2 & -2 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ 0 \\ -4 \end{bmatrix}$$

$$\operatorname{sgn}[\mathbf{Wx}] = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

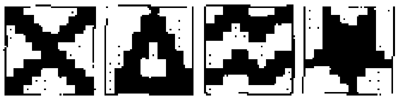
An unstable state. However, it converges to its nearest stable state (-1, 1, -1)^T

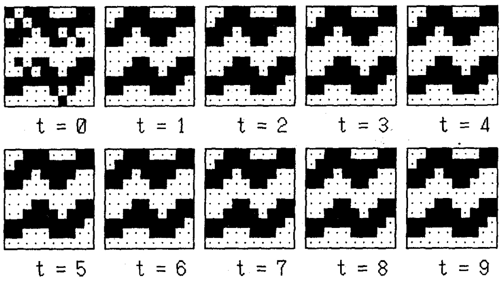
Thus, the synaptic weight matrix can be determined by the two patterns:

$$\mathbf{W} = \frac{1}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & -1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & -1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 0 & -2 & 2 \\ -2 & 0 & -2 \\ 2 & -2 & 0 \end{bmatrix}$$

Computer Experiments





Significance of Hopfield Model

1) The Hopfield model establishes the bridge between various disciplines.
2) The velocity of pattern recalling in Hopfield models is independent on the quantity of patterns stored in the net.

Limitations of Hopfield Model

1) Memory capacity;

The memory capacity is directly dependent on the number of neurons in the network. A theoretical result is

$$p < \frac{N}{2\log N}$$

When N is large, it is approximately

$$p = 0.14N$$

2) Spurious memory;
3) Auto-associative memory;
4) Reinitialization
5) Oversimplification

Problema de Caixeira Viajante

- Buscar um caminho mais curto entre *n* cidades visitando cada cidade somente uma vez e voltando a cidade de partida.
- Um problema clássico de otimização combinatório;
- Algoritmos para encontrar uma solução exato são NP-difíceis

Problema de Caixeira Viajante

$$E = \frac{W_1}{2}\left\{\sum_{i=1}^{n}\left(\sum_{j=1}^{n} x_{ij} - 1\right)^2 + \sum_{j=1}^{n}\left(\sum_{i=1}^{n} x_{ij} - 1\right)^2\right\}$$

$$+ \frac{W_2}{2}\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(x_{k\,j+1} + x_{k\,j-1}\right)x_{ij}\,d_{ij}\right\}$$

$x_{i0} = x_{in}$

$x_{i\,n+1} = x_{i1}$

d_{ij}: distancia entre cidade *i* e *j*

Cidade/Posição	1	2	3	4
1	1	0	0	0
2	0	0	1	0
3	0	0	0	1
4	0	1	0	0

x_{ij}: Output do neurônio (i, j)

N cidades, N^2 neurônios

$$y_{ij}(t+1)=ky_{ij}(t)+\alpha\left\{-W_1\left(\sum_{i\neq j}^{N}x_{ij}(t)+\sum_{k\neq i}^{N}x_{kj}(t)\right)-W_2\left(\sum_{k\neq i}^{N}d_{ik}x_{kj+1}(t)+\sum_{k\neq i}^{N}d_{ik}x_{kj-1}(t)\right)+W_3\right\}$$

$$x_{ij}(t)=\frac{1}{1+e^{-y_{ij}(t)/\varepsilon}}$$

$$x_{ij}(t)=\begin{cases}1 & \mathit{iff} \quad x_{ij}(t)>\sum_{k,l}x_{kl}(t)\Big/N^2\\ 0 & \mathit{otherwise}\end{cases}$$