(Very) Brief Review to Probabilistic Theory

Joint and Conditional Probability

- *P(X=x* and *Y=y)* = *P(x,y)*
- If X and Y are independent then
  P(x,y) = *P(x) P(y)*
- *P(x | y)* is the probability of *x* given *y*
  P(x | y) = *P(x,y) / P(y)*
  P(x,y) = *P(x | y) P(y)*
- If X and Y are independent then
  P(x | y) = *P(x)*

Law of Total Probability, Marginals

Discrete case

$$\sum_x P(x) = 1$$

$$P(x) = \sum_y P(x, y)$$

$$P(x) = \sum_y P(x \mid y) P(y)$$

Continuous case

$$\int p(x)\, dx = 1$$

$$p(x) = \int p(x, y)\, dy$$

$$p(x) = \int p(x \mid y) p(y)\, dy$$

Bayes Rule

$$P(x, y) = P(x \mid y) P(y) = P(y \mid x) P(x)$$

$$\Rightarrow$$

$$P(x \mid y) = \frac{P(y \mid x)\ P(x)}{P(y)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}}$$

Não depende de *x*
Difícil de calcular!

Solução: lei da probabilidade total!

$$P(y) = \sum_x P(y \mid x) P(x)$$

Bayes Rule

$$P(x \mid y) = \frac{P(y \mid x)\ P(x)}{P(y)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}}$$

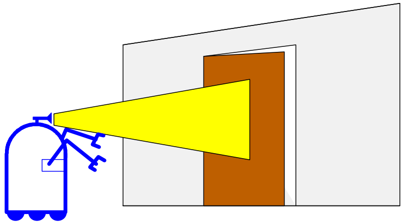
$$P(x \mid y) = \frac{P(y \mid x)\ P(x)}{\sum_x P(y \mid x) P(x)} = \eta\ P(y \mid x)\, P(x)$$

Bayes Rule with Background Knowledge

$$P(x \mid y, z) = \frac{P(y \mid x, z)\ P(x \mid z)}{P(y \mid z)}$$

Simple Example of State Estimation

- Suppose a robot obtains measurement *z*
- What is *P(open|z)?*



Causal vs. Diagnostic Reasoning

- *P(open|z)* is diagnostic.
- *P(z|open)* is causal.
- Often causal knowledge is easier to obtain. **count frequencies!**
- Bayes rule allows us to use causal knowledge:

$$P(open \mid z) = \frac{P(z \mid open)P(open)}{P(z)}$$

Example

- $P(z|open) = 0.6 \qquad P(z|\neg open) = 0.3$
- $P(open) = P(\neg open) = 0.5$

$$P(open \mid z) = ??$$

$$P(x \mid y) = \frac{P(y \mid x)\ P(x)}{P(y)}$$

$$P(x \mid y) = \frac{P(y \mid x)\ P(x)}{\sum_x P(y \mid x) P(x)} = \eta\ P(y \mid x) P(x)$$

Example

- $P(z|open) = 0.6 \qquad P(z|\neg open) = 0.3$
- $P(open) = P(\neg open) = 0.5$

$$P(open \mid z) = \frac{P(z \mid open)P(open)}{P(z \mid open)p(open) + P(z \mid \neg open)p(\neg open)}$$

$$P(open \mid z) = \frac{0.6 \cdot 0.5}{0.6 \cdot 0.5 + 0.3 \cdot 0.5} = \frac{2}{3} = 0.67$$

- *z* raises the probability that the door is open.

Combining Evidence

- Suppose our robot obtains another observation z_2.
- How can we integrate this new information?
- More generally, how can we estimate $P(x \mid z_1 ... z_n)$?

Recursive Bayesian Updating

$$P(x \mid z_1,\ldots,z_n) = \frac{P(z_n \mid x, z_1,\ldots,z_{n-1})\ P(x \mid z_1,\ldots,z_{n-1})}{P(z_n \mid z_1,\ldots,z_{n-1})}$$

Markov assumption: z_n is independent of $z_1,\ldots,z_{n-1}$ if we know *x.*

$$P(x \mid z_1,\ldots,z_n) = \frac{P(z_n \mid x)\ P(x \mid z_1,\ldots,z_{n-1})}{P(z_n \mid z_1,\ldots,z_{n-1})}$$

$$= \eta\ P(z_n \mid x)\ P(x \mid z_1,\ldots,z_{n-1})$$

Actions

- Often the world is **dynamic** since
  - **actions carried out by the robot**,
  - **actions carried out by other agents**,
  - or just the **time** passing by

  change the world.

- How can we **incorporate** such **actions**?

Typical Actions

- The robot **turns its wheels** to move
- The robot **uses its manipulator** to grasp an object
- Plants grow over **time**…

- Actions are **never carried out with absolute certainty**.
- In contrast to measurements, **actions generally increase the uncertainty**.

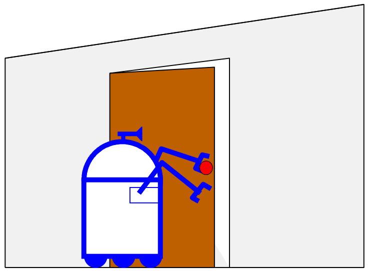
Modeling Actions

- To incorporate the outcome of an action *u* into the current "belief", we use the conditional pdf

$$P(x|u,x')$$

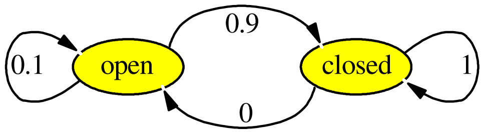
- This term specifies the pdf that **executing *u* changes the state from *x'* to *x*.**

Example: Closing the door



State Transitions

$P(x|u,x')$ for u = "close door":



If the door is open, the action "close door" succeeds in 90% of all cases.

Integrating the Outcome of Actions

Continuous case:

$$P(x|u) = \int P(x|u,x')P(x')dx'$$

Discrete case:

$$P(x|u) = \sum_{x'} P(x|u,x')P(x')$$

Example: The Resulting Belief

$$P(closed \mid u) = \sum_{x'} P(closed \mid u, x')P(x')$$

$$P(open \mid u) = \sum_{x'} P(open \mid u, x')P(x')$$

Example: The Resulting Belief

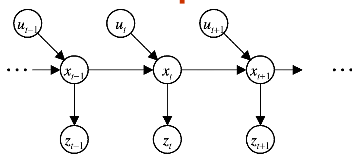
$$\begin{aligned} P(closed \mid u) &= \sum_{x'} P(closed \mid u, x')P(x') \\ &= P(closed \mid u, open)P(open) \\ &\quad + P(closed \mid u, closed)P(closed) \\ &= \frac{9}{10} * \frac{5}{8} + \frac{1}{1} * \frac{3}{8} = \frac{15}{16} \end{aligned}$$

$$\begin{aligned} P(open \mid u) &= \sum_{x'} P(open \mid u, x')P(x') \\ &= P(open \mid u, open)P(open) \\ &\quad + P(open \mid u, closed)P(closed) \\ &= \frac{1}{10} * \frac{5}{8} + \frac{0}{1} * \frac{3}{8} = \frac{1}{16} \\ &= 1 - P(closed \mid u) \end{aligned}$$

Bayes Filters: Framework

- **Given:**
  - Stream of observations *z* and action data *u:*
    $$d_t = \{u_1, z_1 \ldots, u_t, z_t\}$$
  - Sensor model *P(z|x).*
  - Action model *P(x|u,x').*
  - Prior probability of the system state *P(x').*
- **Wanted:**
  - Estimate of the state *X* of a dynamical system.
  - The posterior of the state is also called **Belief**:
    $$Bel(x_t) = P(x_t \mid u_1, z_1 \ldots, u_t, z_t)$$

Markov Assumption



$$p(z_t \mid x_{0:t}, z_{1:t}, u_{1:t}) = p(z_t \mid x_t)$$

$$p(x_t \mid x_{1:t-1}, z_{1:t}, u_{1:t}) = p(x_t \mid x_{t-1}, u_t)$$

Underlying Assumptions

- Static world
- Independent noise
- Perfect model, no approximation errors

Bayes Filters

z = observation
u = action
x = state

$$Bel(x_t) = P(x_t \mid u_1, z_1 \ldots, u_t, z_t)$$

Bayes $$= \eta \; P(z_t \mid x_t, u_1, z_1, \ldots, u_t) \; P(x_t \mid u_1, z_1, \ldots, u_t)$$

Markov $$= \eta \; P(z_t \mid x_t) \; P(x_t \mid u_1, z_1, \ldots, u_t)$$

Total prob. $$= \eta \; P(z_t \mid x_t) \int P(x_t \mid u_1, z_1, \ldots, u_t, x_{t-1}) \; P(x_{t-1} \mid u_1, z_1, \ldots, u_t) \, dx_{t-1}$$

Markov $$= \eta \; P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \; P(x_{t-1} \mid u_1, z_1, \ldots, u_t) \, dx_{t-1}$$

Markov $$= \eta \, P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \, P(x_{t-1} \mid u_1, z_1, \ldots, z_{t-1}) \, dx_{t-1}$$

$$= \eta \; P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \; Bel(x_{t-1}) \, dx_{t-1}$$

$$Bel(x_t) = \eta \; P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \; Bel(x_{t-1}) \, dx_{t-1}$$

Algorithm **Bayes_filter**(*Bel(x),d*):
$\eta = 0$
If *d* is a perceptual data item *z* then
  For all *x* do
    $Bel'(x) = P(z \mid x) Bel(x)$
    $\eta = \eta + Bel'(x)$
  For all *x* do
    $Bel'(x) = \eta^{-1} Bel'(x)$
Else if *d* is an action data item *u* then
  For all *x* do
    $Bel'(x) = \int P(x \mid u, x') \, Bel(x') \, dx'$
Return *Bel'(x)*

Bayes Filters are Familiar!

$$Bel(x_t) = \eta \ P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \ Bel(x_{t-1}) \ dx_{t-1}$$

- Kalman filters
- Particle filters
- Hidden Markov models
- Dynamic Bayesian networks
- Partially Observable Markov Decision Processes (POMDPs)

Summary

- Bayes rule allows us to compute probabilities that are hard to assess otherwise.
- Under the Markov assumption, recursive Bayesian updating can be used to efficiently combine evidence.
- Bayes filters are a probabilistic tool for estimating the state of dynamic systems.